Subelliptic $\mathrm{Spin}_{\mathbb{C}}$ Dirac Operators, IV
Proof of the Relative Index Conjecture

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Abstract

We prove the relative index conjecture, which in turn implies that the set of embeddable deformations of a strictly pseudoconvex CR-structure on a compact 3-manifold is closed in the $\mathcal{C}^\infty$-topology.

1 Proof of the Relative Index Conjecture

In this short paper, which continues the analysis presented in [3], we show how the formula for the relative index between two Szegő projectors $\mathcal{S}_0, \mathcal{S}_1$, defined by two embeddable CR-structures on a contact 3-manifold (Y, H), gives a proof of the relative index conjecture:

Theorem 1. *Let (Y, H) be a compact 3-dimensional co-oriented, contact manifold, and let $\mathcal{S}_0$ be the Szegő projector defined by an embeddable CR-structure with underlying plane field H. There is an M such that for the Szegő projector $\mathcal{S}_1$ defined by any embeddable deformation of the reference structure with the same underlying plane field, we have the upper bound:*

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq M. \tag{1}$$

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Recall that the deformations of a reference CR-structure, $T_b^{0,1}Y$, on (Y,H) are parameterized by

$$\mathrm{Def}(Y,H,\mathcal{S}_0)=\{\Phi\in\mathcal{C}^\infty(Y;\mathrm{Hom}(T_b^{0,1}Y,T_b^{1,0}Y)):\ \|\Phi\|_{L^\infty}<1\},\quad (2)$$

via the prescription:

$${}^{\Phi}T_{b,y}^{0,1}Y=\{\overline{Z}_y+\Phi_y(\overline{Z}_y):\ \overline{Z}_y\in T_{b,y}^{0,1}Y\}.\quad (3)$$

Here and in the sequel we often use the Szegő projector to label a CR-structure. Let $\mathcal{E}\subset\mathrm{Def}(Y,H,\mathcal{S}_0)$ consist of the embeddable deformations, that is, CR-structures arising as pseudoconvex boundaries of complex surfaces. In [2] we showed that if $\mathcal{S}_0$ is Szegő projector defined by the reference CR-structure and $\mathcal{S}_1$ that defined by an embeddable deformation, then the map

$$\mathcal{S}_1:\mathrm{Im}\,\mathcal{S}_0\longrightarrow\mathrm{Im}\,\mathcal{S}_1\quad (4)$$

is a Fredholm operator. $\text{R-Ind}(\mathcal{S}_0,\mathcal{S}_1)$ denotes its Fredholm index, which we call the *relative index*. In the proof of Theorem E in [2] we showed that, for each $m\in\mathbb{N}\cup\{0\}$ and any $\delta>0$, the subsets of $\mathrm{Def}(Y,H,\mathcal{S}_0)$ given by

$$\mathfrak{S}_m^\delta=\{\mathcal{S}_1\in\mathrm{Def}(Y,H,\mathcal{S}_0):\ -\infty<\text{R-Ind}(\mathcal{S}_0,\mathcal{S}_1)\leq m\}\text{ and }\|\Phi\|_{L^\infty}^2\leq\frac{1}{2}-\delta,\quad (5)$$

are closed in the $\mathcal{C}^\infty$-topology. In fact, we show that there is an integer k_0, so that this conclusion holds for a sequence $<\Phi_n>$ converging to Φ in the $\mathcal{C}^{k_0}$-norm.

Combining (1) with Theorem E of [2] we prove:

Corollary 1. *Under the hypotheses of Theorem 1, the set of embeddable deformations of the CR-structure on Y is closed in the $\mathcal{C}^\infty$-topology.*

Proof of the Corollary. Suppose that $<\Phi_n>$ is a sequence of embeddable deformations in $\mathcal{E}\subset\mathrm{Def}(Y,H,\mathcal{S}_0)$ converging to $\Phi\in\mathrm{Def}(Y,H,\mathcal{S}_0)$, in the $\mathcal{C}^\infty$-topology. We first observe that $\|\Phi\|_{L^\infty}<1$.

Let Ψ_1 and Ψ_2 be deformations of the reference structure, with local representations

$$\Psi_j=\psi_jZ\otimes\bar{\omega}.\quad (6)$$

The local representation of Ψ_2 as a deformation of Ψ_1 is given by

$$\psi_{21}=\frac{\psi_2-\psi_1}{1-\overline{\psi_1}\psi_2},\quad (7)$$

see equation (5.5) in [2][I]. We can represent Φ as a deformation of any of the structures in the sequence. From equation (7) it is clear that there an integer N so

that, as deformations of Φ_N, a tail of the sequence and its limit lie in the L^∞-ball in $\mathrm{Def}(Y,H,\mathcal{S}_N)$, centered at 0, of radius $\frac{1}{4}$. Theorem 1 shows that there is an M so that

$$\text{R-Ind}(\mathcal{S}_N,\mathcal{S}_n)\leq M,\ \text{for all } n\in\mathbb{N}. \tag{8}$$

Theorem E from [2] then implies that the limiting structure Φ is also embeddable, completing the proof of the corollary. □

Before proving Theorem 1 we recall the formula for the relative index proved in [3]:

Theorem 2. *Let (Y,H) be a compact 3-dimensional co-oriented, contact manifold, and let $\mathcal{S}_0,\mathcal{S}_1$ be Szegő projectors for embeddable CR-structures with underlying plane field H. Suppose that (X_0,J_0), (X_1,J_1) are strictly pseudoconvex complex manifolds with boundaries $(Y,H,\mathcal{S}_0)$, $(Y,H,\mathcal{S}_1)$, respectively, then*

$$\begin{aligned}\text{R-Ind}(\mathcal{S}_0,\mathcal{S}_1)=&\dim H^{0,1}(X_0,J_0)-\dim H^{0,1}(X_1,J_1)+\\&\frac{\mathrm{sig}[X_0]-\mathrm{sig}[X_1]+\chi[X_0]-\chi[X_1]}{4}.\end{aligned} \tag{9}$$

Here $\mathrm{sig}[X]$ is the signature of the non-degenerate quadratic form,

$$([\alpha],[\beta])\mapsto\int_X\alpha\wedge\beta, \tag{10}$$

defined for $[\alpha],[\beta]\in\widehat{H}^2(X)$, the image of $H^2(X,bX)$ in $H^2(X)$, and $\chi[X]$ is the topological Euler characteristic:

$$\chi[X]=\sum_{j=0}^{4}b_j(X)(-1)^j,\ \text{where } b_j(X)=\dim H_j(X;\mathbb{Q}). \tag{11}$$

Proof of Theorem 1. Let X_1 be a minimal resolution of the normal Stein space with boundary $(Y,H,\mathcal{S}_1)$. It follows from a theorem of Bogomolov and De Oliveira that there is a small perturbation of the complex structure on X_1 making it into a Stein manifold, see [1]. Hence it follows that X_1, with a deformed complex structure, has a strictly plurisubharmonic exhaustion function, and therefore X_1 has the homotopy type of a 2-dimensional CW-complex. Thus expanding the formula in (9) gives:

$$\text{R-Ind}(\mathcal{S}_0,\mathcal{S}_1)=C_0-\dim H^{0,1}(X_1,J_1)-\frac{\mathrm{sig}[X_1]+1-b_1(X_1)+b_2(X_1)}{4}, \tag{12}$$

where C_0 denotes the contribution of the terms from the reference structure:

$$C_0 = H^{0,1}(X_0, J_0) + \frac{\operatorname{sig}[X_0] + \chi(X_0)}{4}. \tag{13}$$

The fact that X_1 is homotopic to a 2-complex implies that $b_1(X_1) \leq b_1(Y)$, see [5]. As $\operatorname{sig}[X_1]$ is the signature of the cup product pairing on $\widehat{H}^2(X_1)$, it is evident that

$$|\operatorname{sig}[X_1]| \leq \dim \widehat{H}^2(X_1) \leq \dim H^2(X_1, bX_1) = b_2(X_1). \tag{14}$$

The last equality is a consequence of the Lefschetz duality theorem. Hence $0 \leq b_2(X_1) + \operatorname{sig}[X_1]$, and therefore

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq C_0 + \frac{b_1(Y) - 1}{4}. \tag{15}$$

This completes the proof of the theorem. $\square$

Remarks on the Ozbagci-Stipsicz Conjecture: Note that

$$\operatorname{sig}[X_1] + b_2(X_2) = 2b_2^+(X_1) + b_2^0(X_1),$$

where $b_2^+(X_1)$ is the dimension of the space on which the pairing in (10) is positive and $b_2^0(X_1)$ is the dimension of the kernel of the map $H^2(X_1, bX_1) \to H^2(X_1)$. A global bound on $|\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)|$, among all Szegő projectors $\mathcal{S}_1$ defined by elements of $\mathcal{E}$, is therefore equivalent to an upper bound for $b_2^+(X_1) + b_2^0(X_1) + \dim H^{0,1}(X_1)$, among all Stein spaces, X_1 filling (Y, H). The existence of an upper bound on $b_2^+(X_1) + b_2^0(X_1)$ was conjectured by Ozbagci and Stipsicz, and proved in some special cases, see [5].

The fact, proved in [2], that $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0$, for sufficiently small deformations shows that, for such deformations:

$$\dim H^{0,1}(X_1) + \frac{2b_2^+(X_1) + b_2^0(X_1)}{4} \leq \\ \dim H^{0,1}(X_0) + \frac{2b_2^+(X_0) + b_2^0(X_0) + b_1(Y) - b_1(X_0)}{4}. \tag{16}$$

In [5] Stipsicz shows that for any Stein filling of (Y, H), we have the estimate $b_2^0(X_1) \leq b_1(Y)$, as well as the existence of a constant $K_{(Y,H)}$ so that

$$b_2^-(X_1) \leq 5b_2^+(X_1) + 2 - K_{(Y,H)} + 2b_1(Y). \tag{17}$$

These estimates, along with (16) prove a "germ" form of the Ozbagci–Stipsicz conjecture: among sufficiently small, embeddable deformations of the CR-structure on the boundary of a strictly pseudoconvex surface, the set of numbers

$$\{b_1(X_1), \sigma(X_1), \chi(X_1)\}$$

is finite. The notion of smallness here depends in a complicated way on the reference CR-structure.

Our results suggest a strategy for proving a lower bound on $\mathrm{R\text{-}Ind}(\mathcal{S}_0, \mathcal{S}_1)$, among deformations Φ with $\|\Phi\|_{L^\infty} < 1 - \epsilon$, for an $\epsilon > 0$. Suppose that no such bound exists, one could then choose a sequence $< \Phi_n > \subset \mathcal{E}$ for which $\mathrm{R\text{-}Ind}(\mathcal{S}_0, \mathcal{S}_n)$ tends to $-\infty$. A contradiction would follow immediately if we could show that $< \Phi_n >$ is bounded in the $\mathcal{C}^{k_0+1}$-norm.

While such an *a priori* bound seems unlikely for the original sequence, it would suffice to replace the sequence $< \Phi_n >$ with a "wiggle-equivalent" sequence. Let M_n denote a projective surface containing $(Y, {}^{\Phi_n}T_b^{0,1}Y)$ as a separating hypersurface, see [4]. An equivalent sequence with better regularity might be obtained by wiggling the hypersurfaces defined by $(Y, {}^{\Phi_n}T_b^{0,1}Y)$ within M_n, perhaps using some sort of heat-flow. After composing the resultant deformations with contact transformations, we might be able to obtain a sequence $< \Phi_n' >$ with $\mathrm{R\text{-}Ind}(\mathcal{S}_0, \mathcal{S}_n') = \mathrm{R\text{-}Ind}(\mathcal{S}_0, \mathcal{S}_n)$ that does satisfy an *a priori* $\mathcal{C}^{k_0+1}$-bound. Such an argument would seem to require an improved understanding of the metric geometry of $\mathrm{Def}(Y, H, \mathcal{S}_0)$, as well as the relationship of an abstract deformation to the local extrinsic geometry of Y as a hypersurface in M_n.

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